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Upper semicontinuity result for the solution mapping of a mixed parametric generalized vector quasiequilibrium problem with moving cones

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Abstract In this paper, we give sufficient conditions for the upper semicontinuity property of the solution mapping of a parametric generalized vector quasiequilibrium problem with mixed relations and moving cones. The main result is proven under the assumption that moving cones have local openness/local closedness properties and set-valued maps are cone-semicontinuous in a sense weaker than the usual sense of semicontinuity. The nonemptiness and the compactness of the solution set are also investigated.

Keywords Equilibrium problem · Moving cone · Openness property · Closedness property · Diagonal quasiconvexity

1 Introduction

In this paper, we are interested in the upper semicontinuity of the solution mapping of the general Problem $(\mathcal{P}_t)$ below, where t is a parameter of a topological space T. In this paper, for any set Y, each subset $\alpha(Y)$ of the product $2^Y \times 2^Y$ is called a relation on 2^Y. In the theory of vector equilibrium problems, we often deal with the case $\alpha(Y) = \alpha_j(Y)$, $j = 1, 2, 3, 4,$

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where Y is a topological vector space and

$$\begin{aligned}\alpha_1(Y) &= \{(a,b) \in 2^Y \times 2^Y : a \not\subset b\},\\ \alpha_2(Y) &= \{(a,b) \in 2^Y \times 2^Y : a \subset b\},\\ \alpha_3(Y) &= \{(a,b) \in 2^Y \times 2^Y : a \cap b \neq \emptyset\},\\ \alpha_4(Y) &= \{(a,b) \in 2^Y \times 2^Y : a \cap b = \emptyset\}\end{aligned}$$

($\emptyset$ being the empty set).

Assume that T, K and E are topological spaces; Y_j, $j = 1, 2, 3, 4$, are topological vector spaces; $A' : T \times E \times K \longrightarrow 2^K$, $A : T \times E \times K \longrightarrow 2^K$, $B : T \times E \times K \longrightarrow 2^E$, $F_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $G_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $C_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $j = 1, 2, 3, 4$, are set-valued maps with nonempty values; and $C_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $j = 1, 2, 3, 4$, are such that for each $(t, z, x, \eta) \in T \times E \times K \times K$, $C_j(t, z, x, \eta)$ is a convex cone of Y_j. For each $j = 1, 4$, and each $(t, z, x, \eta) \in T \times E \times K \times K$, we assume that the interior of the set $C_j(t, z, x, \eta)$, denoted by int $C_j(t, z, x, \eta)$, is nonempty. For each $j = 1, 2, 3, 4$, let $\alpha_j(Y_j)$ be the relation introduced above. For $(t, z, x, \eta) \in T \times E \times K \times K$, let us set

$$\begin{aligned}G'_j(t,z,x,\eta) &= G_j(t,z,x,\eta) + C_j(t,z,x,\eta), \quad j = 2, 3,\\ G'_j(t,z,x,\eta) &= G_j(t,z,x,\eta) + \operatorname{int} C_j(t,z,x,\eta), \quad j = 1, 4.\end{aligned}$$

Consider the following Problem $(\mathcal{P}_t)$, where t is a fixed point of T.

Problem $(\mathcal{P}_t)$: Find a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$ and for all $\eta \in A(t, z_0, x_0)$,

$$(F_j(t, z_0, x_0, \eta), G'_j(t, z_0, x_0, \eta)) \in \alpha_j(Y_j), \quad j = 1, 2, 3, 4.$$

It is worth noticing that we do not assume that the set-valued maps appearing in the four relations in this model are the same. So, in general, all the different relations $\alpha_j(Y)$, $j = 1, 2, 3, 4$, can simultaneously appear in this model. This is a motivation for using the term "mixed" in the title of this paper.

Problem $(\mathcal{P}_t)$ is a general version of several different problems in the theory of equilibrium problems. Before mentioning some special cases of this model, let us note that in this paper we are interested only in those cones of a vector space which contain the origin of this space and do not coincide with the whole space.

(a) If $F_j = G_j \equiv \{0\}$, $j = 2, 3, 4$, then conditions

$$(F_j(t, z_0, x_0, \eta), G'_j(t, z_0, x_0, \eta)) \in \alpha_j(Y_j), \quad j = 2, 3, 4,$$

are automatically satisfied, and hence Problem $(\mathcal{P}_t)$ becomes Problem $(\mathcal{P}_t^1)$ of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$ and, for all $\eta \in A(t, z_0, x_0)$,

$$F_1(t, z_0, x_0, \eta) \not\subset G_1(t, z_0, x_0, \eta) + \operatorname{int} C_1(t, z_0, x_0, \eta).$$

Problem $(\mathcal{P}_t^1)$ was widely investigated in many papers. For a list of several special cases of this model, see [1]. Observe that most special cases of Problem $(\mathcal{P}_t^1)$ were treated under the assumption that $A' = \mathrm{cl} A$ (i.e., each value of A' is the closure of the corresponding value of A), $G_1 \equiv \{0\}$ and C_1 is a constant convex cone. Under this special assumption, the upper and lower semicontinuities of the solution set $\mathcal{S}^1(t)$ of Problem $(\mathcal{P}_t^1)$ were given in [3,4]. For the case where $G_1 \equiv \{0\}$, C_1 is a moving cone, but F_1 is a single-valued map, see [18].

If C_1 is a moving cone depending only on $x \in K$, and $G_1 \equiv \{\varepsilon\}$, where ε is a fixed point of the set $\cap_{x \in K} \text{int}\, C_1(x)$, Problem $(\mathcal{P}_t^1)$ reduces to Problem $(\varepsilon{-}GVEP)$ in [19].

Problem $(GVQEP1)$ in [21] corresponds to Problem $(\mathcal{P}_t^1)$, where $G_1 \equiv \{0\}$ and all set-valued maps involved in Problem $(\mathcal{P}_t^1)$ do not depend on z.

(b) If $F_j = G_j \equiv \{0\}$, $j = 1, 3, 4$, then conditions

$$(F_j(t, z_0, x_0, \eta), G'_j(t, z_0, x_0, \eta)) \in \alpha_j(Y_j), \quad j = 1, 3, 4,$$

are automatically satisfied, and hence Problem $(\mathcal{P}_t)$ becomes Problem $(\mathcal{P}_t^2)$ of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$ and, for all $\eta \in A(t, z_0, x_0)$,

$$F_2(t, z_0, x_0, \eta) \subset G_2(t, z_0, x_0, \eta) + C_2(t, z_0, x_0, \eta).$$

If $C_2 \equiv \{0\}$ and $G_2(t, z, x, \eta) \equiv \widetilde{G}_2(t, z, x, x)$ for all $(t, z, x, \eta) \in T \times E \times K \times K$, where $\widetilde{G}_2(t, z, x, x)$ is some set of Y_2, the semicontinuities of the solution set $\mathcal{S}^2(t)$ of Problem $(\mathcal{P}_t^2)$ were examined in [5,6].

For the existence results in Problem $(\mathcal{P}_t^2)$ with $C_2 \equiv \{0\}$, see [31,32]. If C_2 is a constant convex cone, and if $G_2(t, z, x, \eta) = F_2(t, z, x, x)$ (or $F_2(t, z, x, \eta) = G_2(t, z, x, x)$) for all $(t, z, x, \eta) \in T \times E \times K \times K$, existence results for Problem $(\mathcal{P}_t^2)$ can be found in [23,35,36]. Problem $(GVQEP2)$ in [21,22] corresponds to Problem $(\mathcal{P}_t^2)$, where $G_2 \equiv \{0\}$ and all the maps A, A', B and C_2 do not depend on the variable z.

(c) If $F_j = G_j \equiv \{0\}$, $j = 1, 2, 4$, then conditions

$$(F_j(t, z_0, x_0, \eta), G'_j(t, z_0, x_0, \eta)) \in \alpha_j(Y_j), \quad j = 1, 2, 4,$$

are automatically satisfied, and hence Problem $(\mathcal{P}_t)$ becomes Problem $(\mathcal{P}_t^3)$ of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$ and, for all $\eta \in A(t, z_0, x_0)$,

$$F_3(t, z_0, x_0, \eta) \cap [G_3(t, z_0, x_0, \eta) + C_3(t, z_0, x_0, \eta)] \neq \emptyset.$$

If $C_3 \equiv \{0\}$ and $G_3(t, z, x, \eta) \equiv \widetilde{G}_3(t, z, x, x)$ for all $(t, z, x, \eta) \in T \times E \times K \times K$, where $\widetilde{G}_3(t, z, x, x)$ is some set of Y_3, the semicontinuities of the solution set $\mathcal{S}^3(t)$ of Problem $(\mathcal{P}_t^3)$ were examined in [5,6]. Existence results for Problem $(\mathcal{P}_t^3)$ with $C_3 \equiv \{0\}$ can be found in [31] (see also [32]).

(d) If $F_j = G_j \equiv \{0\}$, $j = 1, 2, 3$, then conditions

$$(F_j(t, z_0, x_0, \eta), G'_j(t, z_0, x_0, \eta)) \in \alpha_j(Y_j), \quad j = 1, 2, 3,$$

are automatically satisfied, and hence Problem $(\mathcal{P}_t)$ becomes Problem $(\mathcal{P}_t^4)$ of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$ and, for all $\eta \in A(t, z_0, x_0)$,

$$F_4(t, z_0, x_0, \eta) \cap [G_4(t, z_0, x_0, \eta) + \text{int}\, C_4(t, z_0, x_0, \eta)] = \emptyset.$$

If $A' = \text{cl}A$, $G_4 \equiv \{0\}$ and $C_4(t, z, x, \eta)$ is a constant convex cone, the upper and lower semicontinuities of the solution set $\mathcal{S}^4(t)$ of Problem $(\mathcal{P}_t^4)$ were investigated in [3,4]. Problem $(GVQEP1)$ in [22] is a special case of Problem $(\mathcal{P}_t^4)$, where $G_4 \equiv \{0\}$ and all set-valued maps A, A', B and C_4 do not depend on z. If C_4 is a moving cone depending only on $x \in K$, and $G_4 \equiv \{\varepsilon\}$, where ε is a fixed point of the set $\cap_{x \in K} \text{int}\, C_4(x)$, Problem $(\mathcal{P}_t^4)$ reduces to Problem $(\varepsilon{-}EVEP)$ in [19].

(e) If $F_j = G_j \equiv \{0\}$, $j = 2, 3$, then conditions

$$(F_j(t, z_0, x_0, \eta), G'_j(t, z_0, x_0, \eta)) \in \alpha_j(Y_j),\ j = 2, 3,$$

are automatically satisfied, and hence Problem $(\mathcal{P}_t)$ becomes Problem $(\mathcal{P}_t^{1,4})$ of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$ and, for all $\eta \in A(t, z_0, x_0)$, we simultaneously have

$$F_1(t, z_0, x_0, \eta) \not\subset G_1(t, z_0, x_0, \eta) + \operatorname{int} C_1(t, z_0, x_0, \eta),$$
$$F_4(t, z_0, x_0, \eta) \cap [G_4(t, z_0, x_0, \eta) + \operatorname{int} C_4(t, z_0, x_0, \eta)] = \emptyset.$$

If we additionally assume that F_j, G_j, $j = 1, 4$, are single-valued, then the last two conditions mean that

$$F_1(t, z_0, x_0, \eta) \notin G_1(t, z_0, x_0, \eta) + \operatorname{int} C_1(t, z_0, x_0, \eta),$$
$$F_4(t, z_0, x_0, \eta) \notin G_4(t, z_0, x_0, \eta) + \operatorname{int} C_4(t, z_0, x_0, \eta).$$

From this discussion it is clear that the symmetric vector quasiequilibrium problem in [12] is a special case of Problem $(\mathcal{P}_t^{1,4})$. Similar arguments show that the symmetric strong vector quasiequilibrium problem in [13] is a special case of Problem $(\mathcal{P}_t)$ with $F_j = G_j \equiv \{0\}$, $j = 1, 4$.

We have seen that our model provides a unified approach to several different equilibrium problems. Another motivation for our model is that it allows us to consider practical problems which cannot be treated by the existing approaches. As an example illustrating this remark, consider a vector version of a game with two players. Let K^i, $i = 1, 2$, be the set of the strategies of the ith player. In the classical game, it is required to find a pair of strategies $(x_0^1, x_0^2) \in K^1 \times K^2$ such that

$$f(x_0^1, x_0^2) \le f(\eta^1, x_0^2), \quad \forall \eta^1 \in K^1,$$
$$f(x_0^1, x_0^2) \le f(x_0^1, \eta^2), \quad \forall \eta^2 \in K^2,$$

where $f : K^1 \times K^2 \longrightarrow \mathbb{R}$ is some real function. In the vector version of the game, the players have different multiobjective goals: it is required to find a pair of strategies $(x_0^1, x_0^2) \in K^1 \times K^2$ such that

$$f_1(x_0^1, x_0^2) \notin f_1(\eta^1, x_0^2) + \operatorname{int} C'_1, \quad \forall \eta^1 \in K^1,$$
$$f_2(x_0^1, x_0^2) \in f_2(x_0^1, \eta^2) + C'_2, \quad \forall \eta^2 \in K^2,$$

where for each $j = 1, 2$, C'_j is a convex cone of a topological vector space Y_j, $f_j : K^1 \times K^2 \longrightarrow Y_j$ is a vector-function, and C'_1 is assumed to have nonempty interior. Thus the desired pair of strategies (x_0^1, x_0^2) is such that x_0^1 is a weak efficient solution of $f_1(\cdot, x_0^2)$ with respect to C'_1, while x_0^2 is a strong solution of $f_2(x_0^1, \cdot)$ with respect to C'_2. In this game, we deal simultaneously with two different kinds of the well-known concepts of solutions in vector optimization; and the game cannot be included as a special case of earlier models in (a),(b),(c) or (d), since each of these models uses only one kind of solutions: weak solutions (in (a),(d)) or strong solutions (in (b),(c)). It is easy to see that this game can be examined in the framework of Problem $(\mathcal{P}_t)$ if we set $F_j = G_j \equiv \{0\}$, $j = 3, 4$, $K = K^1 \times K^2$, and $A(t, x, \eta) = A'(t, x, \eta) = K$, $C_j(t, z, x, \eta) = C'_j$, $F_j(t, z, x, \eta) = f_j(x^1, x^2)$, $j = 1, 2$, $G_1(t, z, x, \eta) = f_1(\eta^1, x^2)$, $G_2(t, z, x, \eta) = f_2(x^1, \eta^2)$ for all $t \in T$, $z \in Z$, $x = (x^1, x^2) \in K^1 \times K^2 := K$, $\eta = (\eta^1, \eta^2) \in K^1 \times K^2 := K$.

Problem $(\mathcal{P}_t)$ is also useful for traffic networks. The well known Wardrop's user principle in traffic networks [37] was extended to the vector case in [38,39] where vector equilibrium flows can be found by solving a vector variational inequality problem. We will see that vector equilibrium flows which will be introduced below can be obtained by solving Problem $(\mathcal{P}_t)$.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a traffic network where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of arcs. Let $\mathcal{I}$ be the given set of the origin-destination pairs (shortly, OD-pairs) and P_i be the given set of paths joining OD-pair $i \in \mathcal{I}$. Let $n := \sum_{i \in \mathcal{I}} |P_i|$ where $|P_i|$ denotes the number of the elements of P_i. For a path $p \in P_i$, denote by $\Phi_p \in \mathbb{R}_+$ the traffic flow on this path and by $\Phi := (\Phi_p, p \in P_i, i \in I) \in \mathbb{R}^n_+$ the vector constructed by these flows, where $\mathbb{R}^n_+$ stands for the nonnegative orthant of $\mathbb{R}^n$. For each path flow vector Φ, define the arc flow vector $\varphi := (\varphi_a, a \in \mathcal{A})$ by

$$\varphi_a = \sum_{i \in \mathcal{I}} \sum_{p \in P_i} \delta_{ap} \Phi_p,$$

where $\delta_{ap} = 1$ if arc $a \in p$ and $\delta_{ap} = 0$ otherwise.

In our traffic network, it is assumed that the demand $d = (d_i, i \in \mathcal{I})$ of the traffic flow is given, and that the traffic flow must satisfy the demand, i.e., $\sum_{p \in P_i} \Phi_p = d_i$ for all $i \in \mathcal{I}$. Let

$$K := \left\{ \Phi : \Phi_p \in \mathbb{R}_+, \sum_{p \in P_i} \Phi_p = d_i, \quad \forall i \in \mathcal{I} \right\}$$

be the set of all such traffic flows.

Let $w_a(\varphi) := (u_a(\varphi), v_a(\varphi)) \in \mathbb{R}^q \times \mathbb{R}^s = \mathbb{R}^l (l = q + s)$ be a vector arc weight on arc a and let $w(\varphi) := (w_a(\varphi), a \in \mathcal{A})$ be the $l \times |\mathcal{A}|$-matrix constructed by these vector arc weights. We define the vector weight $W_p(\Phi)$ on the path $p \in P_i$ as the sum of all the arc weights on this path, i.e.,

$$W_p(\Phi) = \sum_{a \in p} w_a(\varphi) \in \mathbb{R}^l.$$

Clearly,

$$W_p(\Phi) = (U_p(\Phi), V_p(\Phi)),$$

where

$$U_p(\Phi) = \sum_{a \in p} u_a(\varphi) \in \mathbb{R}^q, V_p(\Phi) = \sum_{a \in p} v_a(\varphi) \in \mathbb{R}^s.$$

Consider the $l \times n$-matrix $W(\Phi) = (W_p(\Phi), p \in P_i, i \in \mathcal{I})$. It is not difficult to verify that

$$W(\Phi) = \begin{pmatrix} U(\Phi) \\ V(\Phi) \end{pmatrix},$$

where $U(\Phi) := (U_p(\Phi), p \in P_i, i \in \mathcal{I})$ and $V(\Phi) := (V_p(\Phi), p \in P_i, i \in \mathcal{I})$ are $q \times n$ and $s \times n$-matrices, respectively.

Let $C_1 \subset \mathbb{R}^q$ and $C_2 \subset \mathbb{R}^s$ be convex cones, with $\operatorname{int} C_1 \neq \emptyset$. We say that a path flow vector $\Phi \in K$ is a vector equilibrium flow if

$$\begin{aligned} &\forall i \in \mathcal{I}, \forall p, p' \in P_i : \\ &\quad \Phi_p = 0 \text{ whenever } U_p(\Phi) - U_{p'}(\Phi) \in \operatorname{int} C_1 \text{ or } V_p(\Phi) - V_{p'}(\Phi) \notin -C_2. \end{aligned}$$

This is a generalized version of the Wardrop principle in traffic networks [37]. Roughly speaking, this definition means that, in the vector equilibrium flow Φ, users only choose the path $p \in P_i$ to travel on, if it is the best one in the sense that the following two conditions

$$\begin{aligned} U_{p'}(\Phi) - U_p(\Phi) &\notin -\text{int } C_1, \\ V_{p'}(\Phi) - V_p(\Phi) &\in C_2 \end{aligned}$$

are simultaneously satisfied for each path p' joining the same OD-pair i. The first (resp. the second) of these conditions proves that p is a weak (resp. strong) solution of the vector optimization problem whose objectives are given by the vector-function $U_{p'}(\Phi)$ (resp. $V_{p'}(\Phi)$) of the variable $p' \in P_i$ and the partial ordering C_1 (resp. C_2). Our definition of a vector equilibrium flow differs from the corresponding ones of [38,39] in that, instead of choosing paths with the mentioned above properties, users only choose Pareto optimal paths [38] or weak minimum paths [39] to travel on. The standard arguments used in [38,39] show that $\Phi \in K$ is a vector equilibrium flow if Φ is a solution of the problem of finding a path flow vector $\Phi \in K$ such that for all $\Phi' \in K$, the following two conditions are simultaneously satisfied:

$$\begin{aligned} U(\Phi)(\Phi' - \Phi) &\notin -\text{int } C_1, \\ V(\Phi)(\Phi' - \Phi) &\in C_2. \end{aligned}$$

Obviously, the just mentioned problem is a special case of Problem $(\mathcal{P}_t)$ and hence, our model is useful in studying traffic networks.

This paper is also motivated by a result (see Proposition 3.1 of [24]) saying that, for a set-valued map $C : X \longrightarrow 2^Y$ from a topological space X to a normed space Y such that for all $x \in X$, $C(x)$ is a closed (not necessarily convex) cone, the upper semicontinuity of C at $x_0 \in X$ is equivalent to the fact that $C(x) \subset C(x_0)$ for all x in some neighborhood of x_0. Hence, if we additionally assume that int $C(x) \neq \emptyset$ and $C(x)$ is convex for all x near x_0, then the upper semicontinuity of both $C(\cdot)$ and the complementary map $C^c(\cdot) := Y \setminus C(\cdot)$ at x_0 leads to the assumption that $C(x) = C(x_0)$ for all x in some neighborhood of x_0. This shows that the upper semicontinuity assumption of each of the moving cones $C(\cdot)$, $C^c(\cdot)$ or of both of them in some recent papers (see, e.g., [21,22] and references therein) is a very strong one (at least for the case of normed spaces) and is satisfied only for a very restrictive class of problems. So, when dealing with moving cones we need to replace this assumption by the weaker ones. In this paper, we will use such weaker assumptions, called openness and closedness properties. These assumptions are quite diffferent from those of [33] in that our assumptions are related directly to C_j, while the corresponding conditions in [33] are imposed on the set-valued maps C_j^+ whose values are the nonnegative dual cones of values of C_j.

The aim of the present paper is to give sufficient conditions for the upper semicontinuity, nonemptiness and compactness of the solution set of Problem $(\mathcal{P}_t)$. The reader is referred to [2,3,5,6,8–11,14–20,26–28,40–42] for different aspects of sensitivity analysis, such as the lower semicontinuity, the continuity, the Hölder and Lipschitz continuity, the differentiability..., of solution mappings in several kinds of equilibrium problems and variational inequality problems.

2 Preliminaries

Let X be a topological space. Each subset of X is a topological space with the induced topology. In this paper, neighborhoods of $x \in X$ are understood as open neighborhoods, and they are denoted by $U(x), U_1(x), U_2(x), \ldots$ The symbols cl M and int M are used to denote the closure and interior of M. If M is a subset of a vector space, then co M denotes the convex hull of M.

For a set-valued map $\varphi : T \longrightarrow 2^X$ between topological spaces T and X, we denote by dom φ and gr φ the domain and graph of φ:

$$\text{dom } \varphi = \{t \in T : \varphi(t) \neq \emptyset\},$$
$$\text{gr } \varphi = \{(t, x) \in T \times X : x \in \varphi(t)\}.$$

If $\psi : T \longrightarrow 2^X$ is another set-valued map, then the intersection $\varphi \cap \psi : T \longrightarrow 2^X$ of φ and ψ is defined by $(\varphi \cap \psi)(t) := \varphi(t) \cap \psi(t)$ for all $t \in T$.

We recall the semicontinuity properties of set-valued maps in the usual sense of [7]. Let $\varphi : T \longrightarrow 2^X$ be a set-valued map between topological spaces T and X. Map φ is upper semicontinuous (shortly, usc) at $t_0 \in T$ if, for any open set $\mathcal{N}$ of X with $\mathcal{N} \supset \varphi(t_0)$, there exists a neighborhood $U(t_0)$ of t_0 such that $\mathcal{N} \supset \varphi(t)$, $\forall t \in U(t_0)$. Map φ is lower semicontinuous (shortly, lsc) at $t_0 \in T$ if, for any open set $\mathcal{N}$ of X with $\mathcal{N} \cap \varphi(t_0) \neq \emptyset$, there exists a neighborhood $U(t_0)$ of t_0 such that $\mathcal{N} \cap \varphi(t) \neq \emptyset$, $\forall t \in U(t_0)$. Map φ is usc (resp. lsc) on $T' \subset T$ if $T' \subset$ dom φ and if φ is usc (resp. lsc) at each point $t \in T'$. If gr φ is an open (resp. closed) set in $T \times X$, then we say that φ has open (resp. closed) graph. A map having closed graph is also called a closed map.

In this paper, we need the following notions which are special cases of the more general definitions introduced in [34].

Definition 2.1 Map φ has openness property at $t_0 \in T$ if, for all $x_0 \in \varphi(t_0)$, there exist neighborhoods $U(t_0)$ and $U(x_0)$ such that

$$x \in \varphi(t), \ \forall (t, x) \in U(t_0) \times U(x_0).$$

Map φ has closedness property at t_0 if , for all $x_0 \notin \varphi(t_0)$, there exist neighborhoods $U(t_0)$ and $U(x_0)$ such that

$$x \notin \varphi(t), \ \forall (t, x) \in U(t_0) \times U(x_0).$$

It is easy to check that φ has closedness property at $t_0 \in T$ if and only if φ' has openness property at this point, where $\varphi' : T \longrightarrow 2^X$ is defined by $\varphi'(t) = X \setminus \varphi(t)$ for all $t \in T$.

We say that φ has openness property (resp. closedness property) on a subset $T' \subset T$ if it has openness property (resp. closedness property) at each point $t_0 \in T'$.

Remark 2.1 Observe that φ has open (resp. closed) graph if and only if φ has openness property (resp. closedness property) at each point $t_0 \in$ dom φ. A set-valued map having openness property (resp. closedness property) at a point t_0 may not have open graph (resp. closed graph), but it must have open (resp. closed) value at t_0.

As a direct consequence of the above definitions, we get the following proposition. (For the more general case, see [34].)

Proposition 2.1 *Let $\varphi, \psi : T \longrightarrow 2^X$ be set-valued maps between topological spaces T and X. Let $t_0 \in$ dom$(\varphi \cap \psi)$. If both φ and ψ have openness property (resp. closedness property) at t_0, then $\varphi \cap \psi$ has openness property (resp. closedness property) at t_0.*

The following proposition is easily derived from the proof of Proposition 2 on page 71 of [7].

Proposition 2.2 *Let $\varphi, \psi : T \longrightarrow 2^X$ be set-valued maps between topological spaces T and X, and let $t_0 \in \mathrm{dom}(\varphi \cap \psi)$. Assume that*

(i) *φ has closedness property at t_0.*
(ii) *ψ is usc and compact-valued at t_0.*

Then $\varphi \cap \psi$ is usc and compact-valued at t_0.

We now recall the notions of C-lower semicontinuity and C-upper semicontinuity of set-valued maps introduced in [30].

Definition 2.2 Let $F : X \longrightarrow 2^Y$ be a set-valued map between a topological space X and a topological vector space Y. Let $C : X \longrightarrow 2^Y$ be a map such that, for each $x \in X$, $C(x)$ is a cone. F is called C-lower semicontinuous (shortly, C-lsc) at $x_0 \in X$ if there exists a compact set $W(x_0) \subset C(x_0)$ such that, for any open set $\mathcal{N}$ with $F(x_0) \cap \mathcal{N} \neq \emptyset$, we can find a neighborhood $U(x_0)$ of x_0 such that $F(x) \cap [\mathcal{N} - W(x_0)] \neq \emptyset$ for all $x \in U(x_0)$. This set $W(x_0)$ is called the set associated to the C-lower semicontinuity of F at x_0. F is called C-upper semicontinuous (shortly, C-usc) at $x_0 \in X$ if there exists a compact set $W(x_0) \subset C(x_0)$ such that, for any open set $\mathcal{N} \supset F(x_0)$, we can find a neighborhood $U(x_0)$ of x_0 such that $\mathcal{N} + W(x_0) \supset F(x)$ for all $x \in U(x_0)$. This set $W(x_0)$ is called the set associated to the C-upper semicontinuity of F at x_0. F is called C-lsc (resp. C-usc) on a subset of X if it is C-lsc (resp. C-usc) at each point of this set.

Remark 2.2 Clearly, if F is lsc (resp. usc) at $x_0 \in X$, then it is C-lsc (resp. C-usc) at x_0 for arbitrary set-valued-map C whose values are cones, since we can take $W(x_0) = \{0\}$. A map F which is C-lsc (resp. C-usc) at $x_0 \in X$ may not be lsc (resp. usc) at this point. For examples illustrating this remark, see [30] and see also Example 3.1.

We now recall some concepts and results of [29,31] for later use.

Let $\alpha(Y)$ be a relation on 2^Y where Y is some nonempty set. Let a be a nonempty convex subset of a vector space X. Let $f, c : a \times a \longrightarrow 2^Y$ be maps such that for all $(x, \eta) \in a \times a$, $f(x, \eta)$ and $c(x, \eta)$ are nonempty sets. We say that the pair (f, c) is $\alpha(Y)$-diagonally quasiconvex in the second variable η if, for each finite subset $\{x_i,\ i = 1, 2, \ldots, n\} \subset a$ and each point $x \in \mathrm{co}\,\{x_i,\ i = 1, 2, \ldots, n\}$, there exists a point x_i such that $(f(x, x_i), c(x, x_i)) \in \alpha(Y)$. For characterizations of the $\alpha(Y)$-diagonal quasiconvexity, see [31].

Using arguments similar to those of the proof of [31, Corollary 3.2] we can obtain the following result.

Theorem 2.1 *Let X, Y and Z' be locally convex Hausdorff topological vector spaces; $K \subset X$ and $E' \subset Z'$ be nonempty convex compact subsets; and $\alpha(Y)$ be a relation on 2^Y. Let $A_1' : E' \times K \longrightarrow 2^K$, $A_1 : E' \times K \longrightarrow 2^K$, $B' : E' \times K \longrightarrow 2^{E'}$, $F' : E' \times K \times K \longrightarrow 2^Y$ and $C' : E' \times K \times K \longrightarrow 2^Y$ be set-valued maps with nonempty values. Assume that A_1' is a usc map with closed convex values; A_1 is a lsc map with closed convex values such that $A_1 \subset A_1'$; B' is an acyclic map; and $L_\alpha : E' \times K \longrightarrow 2^K$ is closed and has convex values, where L_α is defined by*

$$L_\alpha(z', \eta) = \{x \in K : (F'(z', x, \eta), C'(z', x, \eta)) \in \alpha(Y)\},\ (z', \eta) \in E' \times K.$$

Assume furthermore that, for each $z' \in E'$, the pair $(F'(z',\cdot,\cdot), C'(z',\cdot,\cdot))$ is $\alpha(Y)$-diagonally quasiconvex in the second variable η. Then there exists a point $(z'_0, x_0) \in E' \times K$ such that $(z'_0, x_0) \in B'(z'_0, x_0) \times A'_1(z'_0, x_0)$ and

$$(F'(z'_0, x_0, \eta), C'(z'_0, x_0, \eta)) \in \alpha(Y), \forall \eta \in A_1(z'_0, x_0).$$

Recall that a set-valued map f between a topological space X and a topological vector space Y is said to be acyclic if it is upper semicontinuous on X and if, for all $x \in X$, $f(x)$ is nonempty, compact and acyclic. Here a topological space is said to be acyclic [25] if all of its reduced Čech homology groups over rationals vanish. Observe that contractible spaces are acyclic; and hence, convex sets and star-shaped sets are acyclic.

3 Main result

In this section, we assume that T, K and E are topological spaces; Y_j, $j = 1, 2, 3, 4$, are topological vector spaces; $A' : T \times E \times K \longrightarrow 2^K$, $A : T \times E \times K \longrightarrow 2^K$, $B : T \times E \times K \longrightarrow 2^E$, $F_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $G_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $C_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $j = 1, 2, 3, 4$, are set-valued maps with nonempty values; and $C_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$, $j = 1, 2, 3, 4$, are such that for each $(t, z, x, \eta) \in T \times E \times K \times K$, $C_j(t, z, x, \eta)$ is a convex cone of Y_j. For each $j = 1, 4$, and each $(t, z, x, \eta) \in T \times E \times K \times K$, we assume that the interior of the set $C_j(t, z, x, \eta)$, denoted by int $C_j(t, z, x, \eta)$, is nonempty. Making use of the definitions of the relations $\alpha_j(Y_j)$, and the sets $G'_j(t, z, x, \eta)$, $j = 1, 2, 3, 4$ (see the Introduction), we can restate Problem $(\mathcal{P}_t)$ as follows.

Problem $(\mathcal{P}_t)$: Find a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t, z_0, x_0) \times A'(t, z_0, x_0)$, and for all $\eta \in A(t, z_0, x_0)$, the following conditions are simultaneously satisfied:

$$\begin{aligned}
&F_1(t, z_0, x_0, \eta) \not\subset G_1(t, z_0, x_0, \eta) + \operatorname{int} C_1(t, z_0, x_0, \eta),\\
&F_2(t, z_0, x_0, \eta) \subset G_2(t, z_0, x_0, \eta) + C_2(t, z_0, x_0, \eta),\\
&F_3(t, z_0, x_0, \eta) \cap [G_3(t, z_0, x_0, \eta) + C_3(t, z_0, x_0, \eta)] \neq \emptyset,\\
&F_4(t, z_0, x_0, \eta) \cap [G_4(t, z_0, x_0, \eta) + \operatorname{int} C_4(t, z_0, x_0, \eta)] = \emptyset.
\end{aligned}$$

Before considering the upper semicontinuity of the solution mapping of Problem $(\mathcal{P}_t)$, we need to establish some lemmas.

Lemma 3.1 *Let $G : X \longrightarrow 2^Y$ be a set-valued map between a topological space X and a topological vector space Y. Let $C : X \longrightarrow 2^Y$ be a set-valued map such that, for each $x \in X$, $C(x)$ is a convex cone. Let $x_0 \in X$.*

(i) *Assume that G is C-usc and compact-valued at x_0 and C has closedness property at x_0. Then the set-valued map $H : X \longrightarrow 2^Y$, defined by*

$$H(x) := G(x) + C(x), \quad x \in X,$$

has closedness property at x_0.

(ii) *Assume that G is C-lsc at x_0 and* int C *has openness property at x_0, where* int C : $X \longrightarrow 2^Y$ *is defined by* (int $C)(x)$ = int $C(x), x \in X$. *Then the set-valued map $H' : X \longrightarrow 2^Y$, defined by*

$$H'(x) := G(x) + \operatorname{int} C(x), \quad x \in X,$$

has openness property at x_0. (*Here* int $C(x)$ *is assumed to be nonempty for each $x \in X$.*)

Proof (i) For every $y_0 \notin H(x_0)$, we have $[y_0 - G(x_0)] \cap C(x_0) = \emptyset$. This implies that

$$[y_0 - G(x_0) - W(x_0)] \cap C(x_0) = \emptyset,$$

where $W(x_0)$ is the compact subset associated to the definition of the C-upper semicontinuity of G at x_0. (Recall that $W(x_0)$ is a subset of the convex cone $C(x_0)$.) Since C has closedness property at x_0 and the set $y_0 - G(x_0) - W(x_0)$ is compact, there exist a neighborhood $U(x_0)$ of x_0 and a neighborhood V of $0_Y \in Y$ such that

$$y \notin C(x), \ \forall (x, y) \in U(x_0) \times [(y_0 + V) - (G(x_0) + W(x_0) - V)].$$

By the C-upper semicontinuity property of G at x_0, we may assume without loss of generality that

$$\forall x \in U(x_0), G(x) \subset G(x_0) + W(x_0) - V.$$

Therefore, for all $x \in U(x_0)$, $y \in y_0 + V$, $y' \in G(x)$, we have

$$y - y' \notin C(x).$$

This shows that $[y - G(x)] \cap C(x) = \emptyset$, i.e., $y \notin G(x) + C(x) = H(x)$. Hence, $y \notin H(x)$ for all $(x, y) \in U(x_0) \times (y_0 + V)$.

(ii) For each $y_0 \in H'(x_0)$, there exists $y_0' \in G(x_0)$ such that $y_0 - y_0' \in \operatorname{int} C(x_0)$. This implies that

$$y_0 - [y_0' - W(x_0)] \subset \operatorname{int} C(x_0) + C(x_0) \subset \operatorname{int} C(x_0),$$

where $W(x_0)$ is the compact subset associated to the definition of the C-lower semicontinuity of G at x_0. (Recall that $W(x_0)$ is a subset of the convex cone $C(x_0)$.) By the openness property of the map int C at x_0 and by the compactness of $W(x_0)$, there exist a neighborhood $U(x_0)$ of x_0 and a neighborhood V of $0_Y \in Y$ such that

$$y \in \operatorname{int} C(x), \ \forall (x, y) \in U(x_0) \times (y_0 + V - [y_0' - W(x_0) - V]).$$

By the C-lower semicontinuity property of G at x_0, we may assume without loss of generality that

$$\forall x \in U(x_0), y_0' \in G(x) + W(x_0) + V.$$

Hence, for all $x \in U(x_0)$, there exists $y' \in G(x)$ such that

$$y' \in y_0' - V - W(x_0).$$

Therefore, for all $(x, y) \in U(x_0) \times (y_0 + V)$, there exists $y' \in G(x)$ such that

$$y - y' \in \operatorname{int} C(x).$$

This shows that $y \in y' + \operatorname{int} C(x) \subset G(x) + \operatorname{int} C(x) = H'(x)$. Hence, $y \in H'(x)$ for all $(x, y) \in U(x_0) \times (y_0 + V)$. □

Making use of the above result, we can prove the following lemma. (In the formulation of the statement (i) of the lemma below, it is assumed that the interior of each value of the set-valued map C is nonempty.)

Lemma 3.2 *Let T, X and Q be topological spaces and let Y be a topological vector space. Let $A : T \times X \longrightarrow 2^Q$ and $F, G : T \times X \times Q \longrightarrow 2^Y$ be set-valued maps with nonempty values. Let $C : T \times X \times Q \longrightarrow 2^Y$ be a set-valued map with nonempty values such that for* *each* $(t, x, q) \in T \times X \times Q$, $C(t, x, q)$ *is a convex cone of* Y. *Let* t_0 *be a point of* T *and let* A *be lsc on* $\{t_0\} \times X$. *Define* $S_i : T \longrightarrow 2^X$, $i = 1, 2, 3, 4$, *by*

$$\begin{aligned}
S_1(t) &= \{x \in X : \exists q \in A(t, x),\ F(t, x, q) \subset G(t, x, q) + \operatorname{int} C(t, x, q)\},\\
S_2(t) &= \{x \in X : \exists q \in A(t, x),\ F(t, x, q) \not\subset G(t, x, q) + C(t, x, q)\},\\
S_3(t) &= \{x \in X : \exists q \in A(t, x),\ F(t, x, q) \cap [G(t, x, q) + C(t, x, q)] = \emptyset\},\\
S_4(t) &= \{x \in X : \exists q \in A(t, x),\ F(t, x, q) \cap [G(t, x, q) + \operatorname{int} C(t, x, q)] \neq \emptyset\},
\end{aligned}$$

where $t \in T$.

(i) *Let* $\operatorname{int} C$ *have openness property on* $\{t_0\} \times X \times Q$*; and let G be* C*-lsc on* $\{t_0\} \times X \times Q$. *If* F *is* C*-usc and compact-valued on* $\{t_0\} \times X \times Q$ *(resp. if* F *is* $(-C)$*-lsc on* $\{t_0\} \times X \times Q$*), then the set-valued map* S_1 *(resp.* S_4*) has openness property at* t_0.

(ii) *Let* C *have closedness property on* $\{t_0\} \times X \times Q$*; and let* G *be* C*-usc and compact-valued on* $\{t_0\} \times X \times Q$. *If* F *is* C*-lsc on* $\{t_0\} \times X \times Q$ *(resp. if* F *is* $(-C)$*-usc and compact-valued on* $\{t_0\} \times X \times Q$*), then the set-valued map* S_2 *(resp.* S_3*) has openness property at* t_0.

Proof (i) For each $(t, x, q) \in T \times X \times Q$, let us set $H'(t, x, q) = G(t, x, q) + \operatorname{int} C(t, x, q)$. We begin by the proof of the openness property of S_1 at t_0.
Assume that $x_0 \in S_1(t_0)$. Then there exists $q_0 \in A(t_0, x_0)$ such that

$$F(t_0, x_0, q_0) \subset H'(t_0, x_0, q_0).$$

Let $W(t_0, x_0, q_0)$ be the compact set associated to the definition of the C-upper semicontinuity property of F at (t_0, x_0, q_0). Since $C(t_0, x_0, q_0)$ is a convex cone, and since $W(t_0, x_0, q_0) \subset C(t_0, x_0, q_0)$, we have

$$F(t_0, x_0, q_0) + W(t_0, x_0, q_0) \subset H'(t_0, x_0, q_0).$$

By the openness property of H' at (t_0, x_0, q_0) (see Lemma 3.1) and by the compactness of the sets $F(t_0, x_0, q_0)$ and $W(t_0, x_0, q_0)$, there exist neighborhoods $U_1(t_0)$ of t_0, $U_1(x_0)$ of x_0, $U_1(q_0)$ of q_0, and a neighborhood V of the origin $0_Y \in Y$ such that

$$\begin{aligned}
&y \in H'(t, x, q),\\
&\quad \forall\ (t, x, q, y) \in U_1(t_0) \times U_1(x_0) \times U_1(q_0)\\
&\qquad \times (F(t_0, x_0, q_0) + W(t_0, x_0, q_0) + V).
\end{aligned} \tag{1}$$

It follows from the C-upper semicontinuity property of F at (t_0, x_0, q_0) that there exist neighborhoods $U_2(t_0) \subset U_1(t_0)$, $U_2(x_0) \subset U_1(x_0)$ and $U_2(q_0) \subset U_1(q_0)$ such that

$$\begin{aligned}
F(t, x, q) \subset F(t_0, x_0, q_0) + W(t_0, x_0, q_0) + V,&\\
\forall (t, x, q) \in U_2(t_0) \times U_2(x_0) \times U_2(q_0).&
\end{aligned} \tag{2}$$

Observe now that $U_2(q_0)$ is an open set having a common point q_0 with $A(t_0, x_0)$. By the lower semicontinuity of A, there exist neighborhoods $U(t_0) \subset U_2(t_0)$ and $U(x_0) \subset U_2(x_0)$ such that

$$A(t, x) \cap U(q_0) \neq \emptyset, \forall (t, x) \in U(t_0) \times U(x_0). \tag{3}$$

Hence, for each $(t, x) \in U(t_0) \times U(x_0)$, by (3) we can find a point $q \in A(t, x)$ such that $q \in U(q_0)$. Since $(t, x, q) \in U(t_0) \times U(x_0) \times U(q_0)$, and since both conditions

(1) and (2) hold, we have

$$F(t,x,q) \subset H'(t,x,q).$$

This shows that there exist neighborhoods $U(t_0)$ and $U(x_0)$ such that

$$x \in S_1(t),\ \forall (t,x) \in U(t_0) \times U(x_0).$$

Therefore, S_1 has openness property at t_0.
Consider now the openness property of S_4 at t_0. Assume that $x_0 \in S_4(t_0)$. Then, we can find a point $q_0 \in A(t_0,x_0)$ such that

$$F(t_0,x_0,q_0) \cap H'(t_0,x_0,q_0) \neq \emptyset,$$

i.e., there exists $y_0 \in F(t_0,x_0,q_0)$ such that $y_0 \in H'(t_0,x_0,q_0)$. Let $W(t_0,x_0,q_0)$ be the compact set associated to the definition of the $(-C)$-lower semicontinuity property of F at (t_0,x_0,q_0). Since $C(t_0,x_0,q_0)$ is a convex cone, and since $W(t_0,x_0,q_0) \subset -C(t_0,x_0,q_0)$, we have

$$y_0 - W(t_0,x_0,q_0) \subset H'(t_0,x_0,q_0).$$

By the openness property of H at (t_0,x_0,q_0) (see Lemma 3.1) and by the compactness of the set $W(t_0,x_0,q_0)$, there exist neighborhoods $U_1(t_0)$ of t_0, $U_1(x_0)$ of x_0, $U_1(q_0)$ of q_0, and a balanced neighborhood V of the origin $0_Y \in Y$ such that

$$y_0 - W(t_0,x_0,q_0) + V \subset H'(t,x,q),\ \forall (t,x,q) \in U_1(t_0) \times U_1(x_0) \times U_1(q_0). \tag{4}$$

From the $(-C)$-lower semicontinuity property of F at (t_0,x_0,q_0), it follows that there exist neighborhoods $U_2(t_0) \subset U_1(t_0)$, $U_2(x_0) \subset U_1(x_0)$ and $U_2(q_0) \subset U_1(q_0)$ such that

$$F(t,x,q) \cap [y_0 - W(t_0,x_0,q_0) + V] \neq \emptyset, \forall (t,x,q) \in U_2(t_0) \times U_2(x_0) \times U_2(q_0). \tag{5}$$

Since $U_2(q_0)$ has a common point q_0 with $A(t_0,x_0)$ and since A is a lsc map, there exist neighborhoods $U(t_0) \subset U_2(t_0)$ and $U(x_0) \subset U_2(x_0)$ such that (3) holds. Hence, for each $(t,x) \in U(t_0) \times U(x_0)$, by (3) we can find a point $q \in A(t,x)$ such that $q \in U(q_0)$. Since $(t,x,q) \in U(t_0) \times U(x_0) \times U(q_0)$ and since (5) holds, there exists a point $y \in F(t,x,q)$ such that $y \in y_0 - W(t_0,x_0,q_0) + V$. From this and from (4), we have

$$y \in H'(t,x,q),$$

and hence, $F(t,x,q) \cap H'(t,x,q) \neq \emptyset$. This shows that there exist neighborhoods $U(t_0)$ and $U(x_0)$ such that

$$x \in S_4(t),\ \forall (t,x) \in U(t_0) \times U(x_0).$$

Therefore, S_4 has openness property at t_0.

(ii) For each $(t,x,q) \in T \times X \times Q$, let us set $H(t,x,q) = G(t,x,q) + C(t,x,q)$. We now prove the openness property of S_2 at t_0. Assume that $x_0 \in S_2(t_0)$. Then there exist $q_0 \in A(t_0,x_0)$ and $y_0 \in F(t_0,x_0,q_0)$ such that

$$y_0 \notin H(t_0,x_0,q_0). \tag{6}$$

Let $W(t_0, x_0, q_0)$ be the compact set associated to the definition of the C-lower semicontinuity property of F at (t_0, x_0, q_0). Since $C(t_0, x_0, q_0)$ is a convex cone, and since $W(t_0, x_0, q_0) \subset C(t_0, x_0, q_0)$, we have from (6) that

$$[y_0 - W(t_0, x_0, q_0)] \cap H(t_0, x_0, q_0) = \emptyset.$$

By the closedness property of H at (t_0, x_0, q_0) (see Lemma 3.1) and by the compactness of the set $W(t_0, x_0, q_0)$, there exist neighborhoods $U_1(t_0)$ of t_0, $U_1(x_0)$ of x_0, $U_1(q_0)$ of q_0, and a neighborhood V of the origin $0_Y \in Y$ such that

$$y \notin H(t, x, q), \quad \forall (t, x, q, y) \in U_1(t_0) \times U_1(x_0) \times U_1(q_0) \times (y_0 - W(t_0, x_0, q_0) + V). \tag{7}$$

By the C-lower semicontinuity property of F at (t_0, x_0, q_0), we can find neighborhoods $U_2(t_0) \subset U_1(t_0)$, $U_2(x_0) \subset U_1(x_0)$ and $U_2(q_0) \subset U_1(q_0)$ such that

$$F(t, x, q) \cap [y_0 - W(t_0, x_0, q_0) + V] \neq \emptyset, \forall (t, x, q) \in U_2(t_0) \times U_2(x_0) \times U_2(q_0). \tag{8}$$

Since $U_2(q_0)$ has a common point q_0 with $A(t_0, x_0)$ and since A is a lsc map, there exist neighborhoods $U(t_0) \subset U_2(t_0)$ and $U(x_0) \subset U_2(x_0)$ such that (3) holds. Hence, for each $(t, x) \in U(t_0) \times U(x_0)$, by (3) we can find a point $q \in A(t, x)$ such that $q \in U(q_0)$. Since $(t, x, q) \in U(t_0) \times U(x_0) \times U(q_0)$ and since (8) holds, we can find a point $y \in F(t, x, q)$ such that $y \in y_0 - W(t_0, x_0, q_0) + V$. From this and from (7), we get

$$y \notin H(t, x, q).$$

This shows that there exist neighborhoods $U(t_0)$ and $U(x_0)$ such that

$$x \in S_2(t), \ \forall (t, x) \in U(t_0) \times U(x_0).$$

Therefore, S_2 has openness property at t_0.
To complete the proof of Lemma 3.2, it remains to establish the openness property of S_3 at t_0. Assume that $x_0 \in S_3(t_0)$. Then, there exists $q_0 \in A(t_0, x_0)$ such that

$$F(t_0, x_0, q_0) \cap H(t_0, x_0, q_0) = \emptyset. \tag{9}$$

Let $W(t_0, x_0, q_0)$ be the compact set associated to the definition of the $(-C)$-upper semicontinuity property of F at (t_0, x_0, q_0). Since $C(t_0, x_0, q_0)$ is a convex cone, and since $W(t_0, x_0, q_0) \subset -C(t_0, x_0, q_0)$, we have from (9) that

$$[F(t_0, x_0, q_0) + W(t_0, x_0, q_0)] \cap H(t_0, x_0, q_0) = \emptyset.$$

By the closedness property of H at (t_0, x_0, q_0) (see Lemma 3.1) and by the compactness of the sets $F(t_0, x_0, q_0)$ and $W(t_0, x_0, q_0)$, there exist neighborhoods $U_1(t_0)$ of t_0, $U_1(x_0)$ of x_0, $U_1(q_0)$ of q_0, and a neighborhood V of the origin $0_Y \in Y$ such that

$$\begin{aligned} y \notin\ & H(t, x, q), \\ & \forall (t, x, q, y) \in U_1(t_0) \times U_1(x_0) \times U_1(q_0) \\ & \times (F(t_0, x_0, q_0) + W(t_0, x_0, q_0) + V). \end{aligned} \tag{10}$$

By the $(-C)$-upper semicontinuity property of F at (t_0, x_0, q_0), we can find neighborhoods $U_2(t_0) \subset U_1(t_0)$, $U_2(x_0) \subset U_1(x_0)$ and $U_2(q_0) \subset U_1(q_0)$ such that

$$\begin{aligned} F(t,x,q) \subset & \; F(t_0,x_0,q_0) \\ & + W(t_0,x_0,q_0) + V, \quad \forall (t,x,q) \in U_2(t_0) \times U_2(x_0) \times U_2(q_0). \quad (11) \end{aligned}$$

Observe now that $U_2(q_0)$ is an open set having a common point q_0 with $A(t_0, x_0)$. By the lower semicontinuity of A, there exist neighborhoods $U(t_0) \subset U_2(t_0)$ and $U(x_0) \subset U_2(x_0)$ such that (3) holds. Hence, for each $(t, x) \in U(t_0) \times U(x_0)$, by (3) we can find a point $q \in A(t, x)$ such that $q \in U(q_0)$. Since $(t, x, q) \in U(t_0) \times U(x_0) \times U(q_0)$, and since both conditions (10) and (11) hold, we have $F(t, x, q) \cap H(t, x, q) = \emptyset$. This shows that there exist neighborhoods $U(t_0)$ and $U(x_0)$ such that

$$x \in S_3(t), \; \forall (t,x) \in U(t_0) \times U(x_0).$$

Therefore, S_3 has openness property at t_0. □

Denote by $\mathcal{S}(t)$ the solution set of Problem $(\mathcal{P}_t)$. We assume that $\mathcal{S}(t_0) \neq \emptyset$ for a fixed point $t_0 \in T$. Let us define the set-valued maps $\psi, \varphi, \varphi_j : T \longrightarrow 2^{E \times K}$, $j = 1, 2, 3, 4$, by setting

$$\begin{aligned} \psi(t) &= \{(z,x) \in E \times K : (z,x) \in B(t,z,x) \times A'(t,z,x)\}, \\ \varphi_j(t) &= \{(z,x) : \forall \eta \in A(t,z,x), (F_j(t,z,x,\eta), G'_j(t,z,x,\eta)) \in \alpha_j(Y_j)\}, \\ \varphi(t) &= \{(z,x) : \forall \eta \in A(t,z,x), (F_j(t,z,x,\eta), G'_j(t,z,x,\eta)) \in \alpha_j(Y_j), j = 1,2,3,4\}, \\ &= \bigcap_{j=1}^{4} \varphi_j(t), \end{aligned}$$

where $t \in T$ and $(z, x) \in E \times K$.

Proposition 3.1 *Assume that $\mathcal{S}(t_0) \neq \emptyset$ for some $t_0 \in T$. Assume that*

(i) *ψ is usc and compact-valued at t_0.*
(ii) *φ has closedness property at t_0.*

Then $\mathcal{S}$ is usc and compact-valued at t_0.

Proof This is a consequence of Proposition 2.2 since $\mathcal{S}(t) = \varphi(t) \cap \psi(t)$ for all $t \in T$. □

The main result of this paper is given in the following theorem.

Theorem 3.1 *Assume that $\mathcal{S}(t_0) \neq \emptyset$ for some $t_0 \in T$. Assume that*

(i) *ψ is usc and compact-valued at t_0 and A is lsc on $\{t_0\} \times E \times K$.*
(ii)$_1$ *The map* int C_1 *has openness property on $\{t_0\} \times E \times K \times K$, F_1 is C_1-usc and compact-valued on $\{t_0\} \times E \times K \times K$, and G_1 is C_1-lsc on $\{t_0\} \times E \times K \times K$.*
(ii)$_2$ *The map C_2 has closedness property on $\{t_0\} \times E \times K \times K$, F_2 is C_2-lsc on $\{t_0\} \times E \times K \times K$, and G_2 is C_2-usc and compact-valued on $\{t_0\} \times E \times K \times K$.*
(ii)$_3$ *The map C_3 has closedness property on $\{t_0\} \times E \times K \times K$, F_3 is $(-C_3)$-usc and compact-valued on $\{t_0\} \times E \times K \times K$, and G_3 is C_3-usc and compact-valued on $\{t_0\} \times E \times K \times K$.*
(ii)$_4$ *The map* int C_4 *has openness property on $\{t_0\} \times E \times K \times K$, F_4 is $(-C_4)$-lsc on $\{t_0\} \times E \times K \times K$, and G_4 is C_4-lsc on $\{t_0\} \times E \times K \times K$.*

Then $\mathcal{S}$ is usc and compact-valued at t_0.

Proof This is a consequence of Lemma 3.2, and Propositions 2.1 and 3.1. Indeed, consider the set-valued maps $\psi, \varphi, \varphi_j : T \longrightarrow 2^{E\times K}$, $j = 1, 2, 3, 4$, defined above. Observe that for $j = 1, 2, 3, 4$, we have

$$\varphi_j(t) = (E \times K) \setminus \varphi'_j(t), \quad t \in T,$$

where

$$\begin{aligned}\varphi'_j(t) \; &= \{(z, x) \in E \times K : \exists \eta \in A(t, z, x),\\ &\quad (F_j(t, z, x, \eta), G'_j(t, z, x, \eta)) \notin \alpha_j(Y_j)\},\ t \in T.\end{aligned}$$

From Lemma 3.2 with $E \times K$ instead of X and K instead of Q, maps φ'_j, $j = 1, 2, 3, 4$, have openness property at t_0, and hence, φ_j, $j = 1, 2, 3, 4$, have closedness property at t_0. By Proposition 2.1, φ has closedness property at t_0. To complete the proof it remains to apply Proposition 3.1. □

Remark 3.1 It is easy to verify that ψ is upper semicontinuous at $t_0 \in T$ if E and K are compact sets, and if A' and B are usc and compact-valued on $\{t_0\} \times E \times K$.

Remark 3.2 In this section, we use the C-lower semicontinuity (resp. the C-upper semicontinuity) of set-valued maps in the sense of Definition 2.2 where the associated set $W(x_0)$ must be a compact subset of the cone $C(x_0)$. It is natural to ask if there exists a special case of C such that our results in this section and in the next section remain valid without the requirement of compactness of $W(x_0)$ in Definition 2.2. The answer to this question is in the affirmative. This is the case where C is a constant set- valued map: $C(x) = C'$ for all x, where C' is a constant convex cone. In this case, we can take $W(x_0) = C'$ and we can modify the corresponding arguments to obtain the desired results. Let us illustrate this remark by considering only the first conclusion of statement (i) of Lemma 3.2. Other conclusions of this lemma and other results of this section and the next section can be proved similarly.

Now we assume that in statement (i) of Lemma 3.2 $C(t, x, q) = C'$ for all $(t, x, q) \in T \times K \times Q$, where $C' \subset Y$ is a convex cone with nonempty interior. We also assume that F is C-lsc (resp. G is C-usc) on $\{t_0\} \times X \times Q$ in the following sense: for any point $(t_0, x_0, q_0) \in \{t_0\} \times X \times Q$ and any open set $\mathcal{N}$ with $F(t_0, x_0, q_0) \cap \mathcal{N} \neq \emptyset$ (resp. $G(t_0, x_0, q_0) \subset \mathcal{N}$) there exist neighborhoods $U(t_0)$, $U(x_0)$ and $U(q_0)$ such that $F(t, x, q) \cap (\mathcal{N} - C') \neq \emptyset$ (resp. $G(t, x, q) \subset \mathcal{N} + C'$) for all $(t, x, q) \in U(t_0) \times U(x_0) \times U(q_0)$. We claim that the first conclusion of statement (i) of Lemma 3.2 remains true under this definition of C-semicontinuities of set-valued maps F and G and under the above assumption of C. We will see that the proof of statement (i) of Lemma 3.2 can be applied if we replace the associated set $W(t_0, x_0, q_0)$ by C'. Assume that x_0 is a point belonging to the set $S_1(t_0)$. Then, there exists $q_0 \in A(t_0, x_0)$ such that

$$F(t_0, x_0, q_0) \subset G(t_0, x_0, q_0) + \operatorname{int} C(t_0, x_0, q_0) =: H'(t_0, x_0, q_0).$$

We claim that the first conclusion of statement (i) of Lemma 3.2 holds if the map H' has openness property at (t_0, x_0, q_0). Indeed, in this case, by the compactness of $F(t_0, x_0, q_0)$, we can find neighborhoods $U_1(t_0)$, $U_1(x_0)$, $U_1(q_0)$ and a neighborhood V of $0_Y \in Y$ such that for all $(t, x, q) \in U_1(t_0) \times U_1(x_0) \times U_1(q_0)$ we have

$$F(t_0, x_0, q_0) + V \subset H'(t, x, q),$$

which implies that

$$F(t_0, x_0, q_0) + V + C' \subset G(t, x, q) + \text{int } C' + C' \subset G(t, x, q) + \text{int } C' \tag{12}$$

since $C(t, x, q) \equiv C'$. Since F is C-usc and compact-valued at (t_0, x_0, q_0), there exist neighborhoods $U_2(t_0) \subset U_1(t_0)$, $U_2(x_0) \subset U_1(x_0)$ and $U(q_0) \subset U_1(q_0)$ such that

$$F(t, x, q) \subset F(t_0, x_0, q_0) + V + C', \quad \forall (t, x, q) \in U_2(t_0) \times U_2(x_0) \times U(q_0). \tag{13}$$

Observe that $U(q_0)$ has a common point q_0 with $A(t_0, x_0)$. Since A is a lsc map, there exist neighborhoods $U(t_0) \subset U_2(t_0)$ and $U(x_0) \subset U_2(x_0)$ such that

$$A(t, x) \cap U(q_0) \neq \emptyset, \ \forall (t, x) \in U(t_0) \times U(x_0).$$

Therefore, for all $(t, x) \in U(t_0) \times U(x_0)$, we can find a point $q \in A(t, x)$ such that $q \in U(q_0)$. Since $(t, x, q) \in U(t_0) \times U(x_0) \times U(q_0)$ and since both conditions (12) and (13) hold, we have

$$F(t, x, q) \subset G(t, x, q) + \text{int } C',$$

i.e.,

$$F(t, x, q) \subset G(t, x, q) + \text{int } C(t, x, q).$$

Since this is true for all $(t, x) \in U(t_0) \times U(x_0)$, we conclude that S_1 has openness property at t_0, as desired.

To complete the proof we need to prove that H' has openness property at (t_0, x_0, q_0). Indeed, for each $y_0 \in H'(t_0, x_0, q_0)$, there exists $y_0' \in G(t_0, x_0, q_0)$ such that

$$y_0 - y_0' \in \text{int } C(t_0, x_0, q_0) = \text{int } C'.$$

By the openness of the set int C' we can find a neighborhood V of the origin $0_Y \in Y$ such that

$$y_0 - y_0' + V + V \subset \text{int } C'.$$

This implies that

$$y_0 + V - [y_0' - V - C'] \subset \text{int } C' + C' \subset \text{int } C'.$$

Since $y_0' \in G(t_0, x_0, q_0) \cap (y_0' - V)$ and since G is C-lsc at (t_0, x_0, q_0), there exist neighborhoods $U(t_0)$ of t_0, $U(x_0)$ of x_0 and $U(q_0)$ of q_0 such that

$$y_0' \in G(t, x, q) + V + C', \forall (t, x, q) \in U(t_0) \times U(x_0) \times U(q_0).$$

Thus, for all $(t, x, q) \in U(t_0) \times U(x_0) \times U(q_0)$, there exists $y' \in G(t, x, q)$ such that

$$y' \in y_0' - V - C'.$$

Therefore, for all $(t, x, q, y) \in U(t_0) \times U(x_0) \times U(q_0) \times (y_0 + V)$, there exists $y' \in G(t, x, q)$ such that

$$y - y' \in \text{int } C(t, x, q).$$

This shows that $y \in y' + \text{int } C(t, x, q) \subset G(t, x, q) + \text{int } C(t, x, q) = H'(t, x, q)$. Hence, $y \in H'(t, x, q)$ for all $(t, x, q, y) \in U(t_0) \times U(x_0) \times U(q_0) \times (y_0 + V)$. Thus H' has openness property at (t_0, x_0, q_0), as required.

Remark 3.3 We have seen that our main result is obtained under the assumption that each of the moving cones C_j has openness/closedness properties and each of the set-valued maps F_j and G_j is cone-semicontinuous in the sense weaker than the usual definition of semicontinuity. To illustrate this remark, we consider the following example, where the set-valued maps F_j, $j = 2, 4$, are not lsc, but they are C_j-lsc; and the moving cones C_j are not usc, but they have openness or closedness properties.

Example 3.1 In this example, we consider Problem $(\mathcal{P}_t^{2,4})$, i.e., Problem $(\mathcal{P}_t)$ where F_j, G_j, C_j and α_j, $j = 1, 3$, are absent. Let $Z = X = \mathbb{R}$, $Y_2 = Y_4 = \mathbb{R}^4$, $E = K = [0, 1]$, $T = [1/2, 1] \subset \mathbb{R}$, and let $\delta \in]0, 1[$. For any $(t, z, x, \eta) \in T \times E \times K \times K$, let us define $A'(t, z, x) = A(t, z, x) = [0, tx]$, $B(t, z, x) = [0, 1 - tz]$, $G_2(t, z, x, \eta) = G_4(t, z, x, \eta) = \{(\delta t, \delta z, \delta x, \delta \eta)\}$,

$$
\begin{aligned}
&C_2(t, z, x, \eta) = \text{cone}\,\{(t, z, x, \eta)\},\\
&C_4(t, z, x, \eta) = \{(t', z', x', \eta') \in \mathbb{R}^4 : tt' + zz' + xx' + \eta\eta' \geq 0\},\\
&F_2(t, z, x, \eta) = \begin{cases} \{(-t, -z, -x, -\eta)\} & \text{if } (z, x, \eta) \neq (0, 0, 0),\\ \{(t, z, x, \eta)\} & \text{if } (z, x, \eta) = (0, 0, 0),\end{cases}\\
&F_4(t, z, x, \eta) = \begin{cases} \{(t, z, x, \eta)\} & \text{if } (z, x, \eta) \neq (0, 0, 0),\\ [(0, 0, 0, 0), (\delta t, 0, 0, 0)] & \text{if } (z, x, \eta) = (0, 0, 0),\end{cases}
\end{aligned}
$$

It is easy to verify that, for each $t_0 \in T$, the solution set $\mathcal{S}^{2,4}(t_0)$ of the above Problem $(\mathcal{P}_{t_0}^{2,4})$ is nonempty since it contains the point $(z_0, x_0) = (0, 0)$. From the definitions of A' and B, it follows that the map ψ is usc and compact-valued at each point $t_0 \in T$. Notice that C_2 has closed graph, and int C_4 has open graph. The map F_2 is not lsc on $T \times E \times K \times K$, but it is C_2-lsc on this set since the set $W_2(t, z, x, \eta)$ associated to the C_2-lower semicontinuity of F_2 at each point $(t, z, x, \eta) \in T \times E \times K \times K$ can be defined as follows: $W_2(t, z, x, \eta) = \{(0, 0, 0, 0), (2t, 2z, 2x, 2\eta)\}$ if $(z, x, \eta) = (0, 0, 0)$, and $W_2(t, z, x, \eta) = \{(0, 0, 0, 0)\}$ if $(z, x, \eta) \neq (0, 0, 0)$. The map F_4 is not lsc on $T \times E \times K \times K$, but it is $(-C_4)$-lsc on this set, where the set $W_4(t, z, x, \eta)$ associated to the $(-C_4)$-lower semicontinuity of F_4 at each point $(t, z, x, \eta) \in T \times E \times K \times K$ can be defined as follows: $W_4(t, z, x, \eta) = \{(0, 0, 0, 0)\} \cup [(-t, -z, -x, -\eta), (\delta t - t, -z, -x, -\eta)]$ if $(z, x, \eta) = (0, 0, 0)$, and $W_4(t, z, x, \eta) = \{(0, 0, 0, 0)\}$ if $(z, x, \eta) \neq (0, 0, 0)$. Thus, for each $t_0 \in T$, all the Assumptions (i), $(\text{ii})_2$ and $(\text{ii})_4$ of Theorem 3.1 are satisfied. By Theorem 3.1, the solution map $\mathcal{S}^{2,4}$ of Problem $(\mathcal{P}_t^{2,4})$ is usc and compact-valued on T.

4 Existence and compactness of solutions

This section is devoted to the solution existence in Problem $(\mathcal{P}_t)$ with $t = t_0$. In other words, we will be interested in the sufficient conditions under which $\mathcal{S}(t_0) \neq \emptyset$. Before establishing an existence result for Problem $(\mathcal{P}_t)$ at $t = t_0 \in T$, we consider the following lemma.

Lemma 4.1 *Let E and K be topological spaces, and Y be a topological vector space. Assume that $F, G : E \times K \times K \longrightarrow 2^Y$ are maps with nonempty values, and $C : E \times K \times K \longrightarrow 2^Y$ is a map such that, for each $(z, x, \eta) \in E \times K \times K$, $C(z, x, \eta)$ is a convex cone. Define*

$L_i : E \times K \longrightarrow 2^K, i = 1, 2, 3, 4$, *by*

$$L_1(z, \eta) = \{x \in K : F(z, x, \eta) \not\subset G(z, x, \eta) + \operatorname{int} C(z, x, \eta)\},$$
$$L_2(z, \eta) = \{x \in K : F(z, x, \eta) \subset G(z, x, \eta) + C(z, x, \eta)\},$$
$$L_3(z, \eta) = \{x \in K : F(z, x, \eta) \cap [G(z, x, \eta) + C(z, x, \eta)] \neq \emptyset\},$$
$$L_4(z, \eta) = \{x \in K : F(z, x, \eta) \cap [G(z, x, \eta) + \operatorname{int} C(z, x, \eta)] = \emptyset\},$$

where $(z, \eta) \in E \times K$.

(i) *Let* int C *have open graph and for each* $(z, x, \eta) \in E \times K \times K$, int $C(z, x, \eta) \neq \emptyset$*; and let* G *be* C*-lsc on* $E \times K \times K$*. If* F *is* C*-usc and compact-valued on* $E \times K \times K$ *(resp. if* F *is* $(-C)$*-lsc on* $E \times K \times K$*), then the set-valued map* L_1 *(resp.* L_4*) has closed graph.*

(ii) *Let* C *have closed graph, and let* G *be* C*-usc and compact-valued on* $E \times K \times K$*. If* F *is* C*-lsc on* $E \times K \times K$ *(resp. if* F *is* $(-C)$*-usc and compact-valued on* $E \times K \times K$*), then the set-valued map* L_2 *(resp.* L_3*) has closed graph.*

Proof Let $T' = E \times K$ and $X = Q = K$. For $t' = (z, \eta) \in T' = E \times K$, $x \in K$ and $q \in Q$, we set

$$F'(t', x, q) = F(z, q, \eta),\ G'(t', x, q) = G(z, q, \eta),\ C'(t', x, q) = C(z, q, \eta),\ A'_1(t', x) = \{x\}.$$

Applying statement (i) of Lemma 3.2 with T', F', G', C' and A'_1 instead of T, F, G, C and A, we derive that the set-valued map $S'_1 : E \times K \longrightarrow 2^K$, defined by

$$S'_1(z, \eta) = \{x \in K : F(z, x, \eta) \subset G(z, x, \eta) + \operatorname{int} C(z, x, \eta)\},$$

has openness property at any point $(z_0, \eta_0) \in E \times K$. This means that S'_1 has open graph. To complete the proof of the closedness of the graph of L_1, it remains to observe that the graph of L_1 is exactly the complement of the graph of S'_1. We delete the similar proof of the closedness of the graph of L_j, $j = 2, 3, 4$. □

In this section, we consider Problem $(\mathcal{P}_{t_0})$ under the additional assumption that E and K are nonempty compact convex subsets of locally convex Hausdorff topological vector spaces Z and X, respectively. Let $A' : T \times E \times K \longrightarrow 2^K$, $A : T \times E \times K \longrightarrow 2^K$, $B : T \times E \times K \longrightarrow 2^E$, $F_j, G_j : T \times E \times K \times K \longrightarrow 2^Y$, $j = 1, 2, 3, 4$, be as in the previous section. For $j = 1, 2, 3, 4$, let $C_j : T \times E \times K \times K \longrightarrow 2^{Y_j}$ be set-valued maps such that, for each $(t, z, x, \eta) \in T \times E \times K \times K$, $C_j(t, z, x, \eta)$ is a convex cone. We also assume that the interior of each value of set-valued maps C_1 and C_4 is nonempty. For each $j = 1, 2, 3, 4$, and each $(t, z, x, \eta) \in T \times E \times K \times K$, let $\alpha_j(Y_j)$ and $G'_j(t, z, x, \eta)$ be as in the Introduction.

For $j = 1, 2, 3, 4$, let us introduce the set-valued map $L_{\alpha_j} : T \times E \times K \longrightarrow 2^K$, defined by

$$L_{\alpha_j}(t, z, \eta) = \{x \in K : (F_j(t, z, x, \eta), G'_j(t, z, x, \eta)) \in \alpha_j(Y_j)\},\ (t, z, \eta) \in T \times E \times K.$$

We set $Y = \prod_{j=1}^{4} Y_j$, and consider the following subset $\alpha(Y)$ of the product $2^Y \times 2^Y$:

$$\alpha(Y) = \left\{ (a, b) \in 2^Y \times 2^Y : a = \prod_{j=1}^{4} a_j,\ b = \prod_{j=1}^{4} b_j, \right.$$
$$\left. a_j, b_j \subset Y_j,\ (a_j, b_j) \in \alpha_j(Y_j),\ j = 1, 2, 3, 4 \right\}.$$

Let us define the set-valued maps $F, G' : T \times E \times K \times K \longrightarrow 2^Y$ as follows

$$F(t, z, x, \eta) = \prod_{j=1}^{4} F_j(t, z, x, \eta),$$

$$G'(t, z, x, \eta) = \prod_{j=1}^{4} G'_j(t, z, x, \eta).$$

For fixed $t_0 \in T$, $j = 1, 2, 3, 4$, let us write C_{j,t_0} instead of $C_j(t_0, \cdot, \cdot, \cdot)$. Thus, the set-valued map $C_{j,t_0} : E \times K \times K \longrightarrow 2^{Y_j}$ is defined by setting

$$C_{j,t_0}(z, x, \eta) = C_j(t_0, z, x, \eta),\ (z, x, \eta) \in E \times K \times K.$$

The set-valued maps $F_{j,t_0}, G_{j,t_0}, G'_{j,t_0}, L_{\alpha_j,t_0}, F_{t_0}, G_{t_0}, G'_{t_0}, A'_{t_0}, A_{t_0}$, and B_{t_0} are defined similarly.

Theorem 4.1 *Let X, Y and Z be locally convex Hausdorff topological vector spaces, and let E and K be nonempty compact convex subsets of Z and X, respectively. Assume that*

- (i) *A'_{t_0} is a usc map with closed convex values; A_{t_0} is a lsc map with closed convex values such that $A_{t_0} \subset A'_{t_0}$; and B_{t_0} is an acyclic map.*
- $(ii)_1$ int C_{1,t_0} *has open graph, F_{1,t_0} is C_{1,t_0}-usc and compact-valued on $E \times K \times K$, and G_{1,t_0} is C_{1,t_0}-lsc on $E \times K \times K$.*
- $(ii)_2$ *C_{2,t_0} has closed graph, F_{2,t_0} is C_{2,t_0}-lsc on $E \times K \times K$, and G_{2,t_0} is C_{2,t_0}-usc and compact-valued on $E \times K \times K$.*
- $(ii)_3$ *C_{3,t_0} has closed graph, F_{3,t_0} is $(-C_{3,t_0})$-usc and compact-valued on $E \times K \times K$, and G_{3,t_0} is C_{3,t_0}-usc and compact-valued on $E \times K \times K$.*
- $(ii)_4$ int C_{4,t_0} *has open graph, F_{4,t_0} is $(-C_{4,t_0})$-lsc on $E \times K \times K$, and G_{4,t_0} is C_{4,t_0}-lsc on $E \times K \times K$.*
- (iii) *For $j = 1, 2, 3, 4, L_{\alpha_j,t_0}$ has convex values, and for each $z \in E$, the pair $(F_{t_0}(z, \cdot, \cdot), G'_{t_0}(z, \cdot, \cdot))$ is $\alpha(Y)$-diagonally quasiconvex in the second variable η.*

Then $\mathcal{S}(t_0) \neq \emptyset$, i.e, there exists a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(t_0, z_0, x_0) \times A'(t_0, z_0, x_0)$ and

$$(F_j(t_0, z_0, x_0, \eta), G'_j(t_0, z_0, x_0, \eta)) \in \alpha_j(Y_j),\ \forall \eta \in A(t_0, z_0, x_0), \forall j = 1, 2, 3, 4.$$

Moreover, $\mathcal{S}(t_0)$ is a nonempty compact set.

Proof Consider the set-valued map $L_{\alpha,t_0} : E \times K \longrightarrow 2^K$, defined by

$$L_{\alpha,t_0}(z, \eta) = \{x \in K : (F_{t_0}(z, x, \eta), G'_{t_0}(z, x, \eta)) \in \alpha(Y)\},\ (z, \eta) \in E \times K.$$

Observe that for all $(z, \eta) \in E \times K$,

$$L_{\alpha,t_0}(z, \eta) = \bigcap_{j=1}^{4} L_{\alpha_j,t_0}(z, \eta),$$

where

$$L_{\alpha_j,t_0}(z, \eta) = \{x \in K : (F_{j,t_0}(z, x, \eta), G'_{j,t_0}(z, x, \eta)) \in \alpha_j(Y_j)\},\quad j = 1, 2, 3, 4.$$

By Lemma 4.1, L_{α_j,t_0}, $j = 1, 2, 3, 4$, are closed maps, and hence, L_{α,t_0} is a closed map. On the other hand, from assumption (iii) we see that L_{α,t_0} has convex values. Thus, L_{α,t_0}

is a closed map with convex values. Applying Theorem 2.1, we obtain the nonemptiness of $\mathcal{S}(t_0)$. Consider now the compactness of this set. Observe that

$$\begin{aligned}\mathcal{S}(t_0) := \{(z,x) \in B_{t_0}(z,x) \times A'_{t_0}(z,x) : \forall \eta \in A_{t_0}(z,x),\\ (F_{j,t_0}(z,x,\eta), G'_{j,t_0}(z,x,\eta)) \in \alpha_j(Y_j),\ j{=}1,2,3,4\},\end{aligned}$$

is the solution set of Problem $(\mathcal{P}_t)$ with $t = t_0$. For each $j = 1, 2, 3, 4$, we set

$$M'_j = \{(z,x) \in E \times K : \forall \eta \in A_{t_0}(z,x), (F_{j,t_0}(z,x,\eta), G'_{j,t_0}(z,x,\eta)) \in \alpha_j(Y_j)\}.$$

It is clear that that $\mathcal{S}(t_0) = M \cap M'$, where

$$\begin{aligned}M &= \{(z,x) \in E \times K : (z,x) \in B_{t_0}(z,x) \times A'_{t_0}(z,x)\},\\ M' &= \bigcap_{j=1}^{4} M'_j.\end{aligned}$$

It is easy to see that M is a compact set. So to prove the compactness of $\mathcal{S}(t_0)$, it remains to show that M' is compact. Observe that, for each $j = 1, 2, 3, 4$, the set M'_j is closed in $E \times K$. Indeed, M'_1 is the complement of the set

$$\begin{aligned}M_1 \ = \{(z,x) \in E \times K : \exists \eta \in A_{t_0}(z,x),\\ F_{j,t_0}(z,x,\eta) \subset G_{j,t_0}(z,x,\eta) + \operatorname{int} C_{j,t_0}(z,x,\eta)\}.\end{aligned}$$

Applying Lemma 3.2 with $X = E \times K$ and $Q = K$, we can infer that M_1 is open in $E \times K$. Thus, M'_1 is closed in $E \times K$. Similarly, we can use Lemma 3.2 to prove that M'_j, $j = 2, 3, 4$, are closed in $E \times K$. Therefore M' is closed in $E \times K$ and hence, it is compact since $E \times K$ is a compact set. □

Remark 4.1 This remark is related to condition (iii) of Theorem 4.1. Sufficient conditions for the convexity of L_{α_j,t_0}, $j = 1, 2, 3, 4$, are given in Proposition 2.3 of [32]. From the definition of $\alpha(Y)$, it is easy to verify that the pair $(F_{t_0}(z,\cdot,\cdot), G'_{t_0}(z,\cdot,\cdot))$ is $\alpha(Y)$-diagonally quasi-convex in the second variable η if and only if for each finite set $\{\eta_i,\ i = 1, 2, \dots, n\} \subset K$ and each point $x \in \operatorname{co}\{\eta_i,\ i = 1, 2, \dots, n\}$, there exists an index $i \in \{1, 2, \dots, n\}$ such that

$$(F_{j,t_0}(z,x,\eta_i), G'_{j,t_0}(z,x,\eta_i)) \in \alpha_j(Y_j),\ j = 1, 2, 3, 4.$$

Remark 4.2 In view of Remark 3.2 we see that the C_j-semicontinuities of F_j and G_j in Theorems 3.1 and 4.1 can be weakened if C_j is a constant convex cone.

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